THE HOMOTOPY CATEGORY OF COMPLEXES OF PROJECTIVE MODULES

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Abstract. The homotopy category of complexes of projective left-modules over any reasonably nice ring is proved to be a compactly generated triangulated category, and a duality is given between its subcategory of compact objects and the finite derived category of right-modules.

0. Introduction

The last decade has seen compactly generated triangulated categories rise to prominence. Triangulated categories go back to Puppe and Verdier, but only later developments have made it clear that the compactly generated ones are particularly useful. For instance, they allow the use of the Brown Representability Theorem and the Thomason Localization Theorem, both proved by Neeman in [4]. There are also results by many other authors to support the case.

The standard examples of compactly generated triangulated categories are the stable homotopy category of spectra and the derived category of a ring. Indeed, many analogies between these two cases are captured by their common structure of compactly generated triangulated category, and this allows the transfer of methods and ideas back and forth.

This paper adds to the collection of compactly generated triangulated categories by showing that if A is a reasonably nice ring, then the homotopy category of complexes of projective A-left-modules, $\mathsf{K}(\mathsf{Pro}\,A)$, is compactly generated.

This may seem slightly surprising in view of [5, app. E.3] which shows that the homotopy category of complexes of all $\mathbb{Z}$-modules, $\mathsf{K}(\mathsf{Mod}\,\mathbb{Z})$, is not even well generated, a weaker notion than compactly generated. However, not only is $\mathsf{K}(\mathsf{Pro}\,A)$ compactly generated; the subcategory of compact objects, $\mathsf{K}(\mathsf{Pro}\,A)^{\mathrm{c}}$, is very nice, in that it is dual to the

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finite derived category of A-right-modules, $\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$, whose objects are complexes with bounded cohomology consisting of finitely presented modules. My proofs of these statements work when A is coherent and satisfies that each flat A-left-module has finite projective dimension.

Most rings encountered in nature, such as noetherian rings, are coherent. The condition that each flat A-left-module has finite projective dimension would appear less standard, but is in fact satisfied by large classes of rings such as noetherian commutative rings of finite Krull dimension ([6, Seconde partie, cor. (3.2.7)]), left-perfect rings ([1, thm. P]), and right-noetherian algebras which admit a dualizing complex ([3]).

The last of these cases includes many non-commutative algebras (see [7] and [8]), among them noetherian complete semi-local PI algebras ([7, cor. 0.2]) and filtered algebras whose associated graded algebras are connected and noetherian and either PI, graded FBN, or with enough normal elements ([8, cor. 6.9]).

It is worth noting that if A has finite left and right global dimension, then there is nothing new in my results. In this case, $\mathsf{K}(\mathsf{Pro}\,A)$ is equivalent to $\mathsf{D}(A)$, the derived category of A-left-modules, so $\mathsf{K}(\mathsf{Pro}\,A)$ is compactly generated. Moreover, the subcategory of compact objects $\mathsf{K}(\mathsf{Pro}\,A)^{\mathrm{c}}$ is equivalent to $\mathsf{D}(A)^{\mathrm{c}}$, the subcategory of compact objects of $\mathsf{D}(A)$, and when A has finite global dimension, $\mathsf{D}(A)^{\mathrm{c}}$ is well known just to be the finite derived category $\mathsf{D}^{\mathrm{f}}(A)$, which is again dual to $\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$ under the functor $\mathrm{RHom}_A(-,A)$.

However, my results work for many rings which do not have finite global dimension.

1. Compact objects

Setup 1.1. In this section, A is a right-coherent ring.

Construction 1.2. Let M be a finitely presented A-left-module. This means that there is an exact sequence of A-left-modules $Q_1 \to Q_0 \to M \to 0$ where Q_0 and Q_1 are finitely generated projective A-left-modules.

Hence there is an exact sequence of A-right-modules $0 \to M^* \to Q_0^* \to Q_1^*$, where $(-)^*$ denotes the functor $\mathrm{Hom}(-,A)$ which dualizes with respect to A.

Here Q_0^* and Q_1^* are finitely generated projective A-right-modules. As M^* is the kernel of a homomorphism between them and as A is right-coherent, it follows that M^* is finitely presented. Hence M^* has a projective resolution P consisting of finitely generated projective A-right-modules.

Viewing M^* as a complex concentrated in degree zero, there is a canonical quasi-isomorphism

$$P \xrightarrow{\pi} M^*.$$

There is also a canonical homomorphism $M \xrightarrow{\mu} M^{**}$ which I will view as a chain map of complexes concentrated in degree zero, and so I can consider

$$M \xrightarrow{\mu} M^{**} \xrightarrow{\pi^*} P^*.$$

Lemma 1.3. *If Q is a projective A-left-module, then*

$$\mathrm{Hom}_A(P^*, Q) \xrightarrow{\mathrm{Hom}_A(\pi^*\mu, Q)} \mathrm{Hom}_A(M, Q)$$

is a quasi-isomorphism.

Proof. As Q is projective, it is a direct summand in a free module, so it is enough to prove the lemma when Q is free. But both P^* and M consist of finitely presented modules so when Q is free, and so has the form $\coprod A$, then the coproduct can be moved outside the Hom's, and so it is enough to prove the lemma for $Q = A$.

There is a commutative diagram

$$\begin{tikzcd}
P \arrow[rr, "\pi"] \arrow[d, "p"'] & & M^* \arrow[d, equal] \\
P^{**} \arrow[r, "\pi^{**}"'] & M^{***} \arrow[r, "\mu^*"'] & M^*
\end{tikzcd}$$

where p is the canonical chain map. Since P consists of finitely generated projective modules, p is an isomorphism. Also, π is a quasi-isomorphism by construction, so the diagram shows that the composition $\mu^*\pi^{**}$ is a quasi-isomorphism.

That is, the chain map

$$\mu^*\pi^{**} = (\pi^*\mu)^* = \mathrm{Hom}_A(\pi^*\mu, A)$$

is a quasi-isomorphism, and this proves the lemma in the case $Q = A$ as desired. □

Lemma 1.4. *If Q is a complex of projective A-left-modules, then*

$$\mathrm{Hom}_A(P^*, Q) \xrightarrow{\mathrm{Hom}_A(\pi^*\mu, Q)} \mathrm{Hom}_A(M, Q)$$

is a quasi-isomorphism.

Proof. The chain map $M \xrightarrow{\pi^*\mu} P^*$ can be completed to a distinguished triangle

$$M \xrightarrow{\pi^*\mu} P^* \longrightarrow C \longrightarrow$$

in the homotopy category of complexes of A-left-modules, $\mathsf{K}(\mathsf{Mod}\,A)$. Here C is bounded to the left because both M and P^* are bounded to the left. This induces a distinguished triangle

$$\mathrm{Hom}_A(C,Q) \longrightarrow \mathrm{Hom}_A(P^*,Q) \xrightarrow{\mathrm{Hom}_A(\pi^*\mu,Q)} \mathrm{Hom}_A(M,Q) \longrightarrow$$

which shows that the chain map in the lemma is a quasi-isomorphism if and only if the complex $\mathrm{Hom}_A(C,Q)$ is exact.

Now, if the complex Q is just a single projective module placed in degree zero, then the lemma follows from lemma 1.3. So in this case, $\mathrm{Hom}_A(C,Q)$ must be exact.

Hence C is a complex bounded to the left for which the complex $\mathrm{Hom}_A(C,Q)$ is exact when Q is a single projective module placed in degree zero. But then it is classical that $\mathrm{Hom}_A(C,Q)$ is exact when Q is any complex of projective modules. Indeed, this follows from an argument analogous to the one which shows that if X is a complex bounded to the left which is exact and I is any complex of injective modules, then $\mathrm{Hom}_A(X,I)$ is exact. □

As indicated in the introduction, the category of projective A-left-modules is denoted $\mathsf{Pro}(A)$, and the corresponding homotopy category of complexes is denoted $\mathsf{K}(\mathsf{Pro}\,A)$. So $\mathsf{K}(\mathsf{Pro}\,A)$ has as objects all complexes of projective A-left-modules, and as morphisms it has homotopy classes of chain maps.

Lemma 1.5. *For each finitely presented A-left-module M, there is a natural equivalence*

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(P^*,-) \simeq \mathrm{H}^0\mathrm{Hom}_A(M,-)$$

of functors on $\mathsf{K}(\mathsf{Pro}\,A)$.

Proof. I have

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(P^*,-) \simeq \mathrm{H}^0\mathrm{Hom}_A(P^*,-) \simeq \mathrm{H}^0\mathrm{Hom}_A(M,-)$$

as functors on $\mathsf{K}(\mathsf{Pro}\,A)$, where the first $\simeq$ is classical and the second $\simeq$ is by lemma 1.4. □

Proposition 1.6. *For each finitely presented A-left-module M, the complex P^* from construction 1.2 is a compact object of $\mathsf{K}(\mathsf{Pro}\,A)$.*

Proof. This is clear from lemma 1.5, since the functor $\mathrm{H}^0\mathrm{Hom}_A(M,-)$ respects set indexed coproducts because M is finitely presented. □

2. Compact generators

Setup 2.1. In this section, A is a coherent ring (that is, it is both left- and right-coherent) for which each flat A-left-module has finite projective dimension.

Remark 2.2. Note that there is an integer N so that the projective dimension of each flat A-left-module F satisfies $\operatorname{pd} F \leq N$. For otherwise, if there were flat A-left-modules of arbitrarily high, finite projective dimension, then the coproduct of such modules would be a flat module of infinite projective dimension.

Construction 2.3. For each finitely presented A-left-module M, take the complex P^* from construction 1.2, and consider the collection of all suspensions $\Sigma^i P^*$.

There is only a set (as opposed to a class) of isomorphism classes of such modules M, so there is also only a set of isomorphism classes in $\mathsf{K}(\mathsf{Pro}\, A)$ of complexes of the form $\Sigma^i P^*$. Let the set $\mathcal{G}$ consist of one object from each such isomorphism class.

Theorem 2.4. *The category $\mathsf{K}(\mathsf{Pro}\, A)$ is a compactly generated triangulated category with $\mathcal{G}$ as a set of compact generators.*

Proof. Each complex P^* is a compact object of $\mathsf{K}(\mathsf{Pro}\, A)$ by proposition 1.6, so the same holds for each complex $\Sigma^i P^*$ in $\mathcal{G}$. It remains to show that $\mathcal{G}$ is a set of generators. So suppose that Q in $\mathsf{K}(\mathsf{Pro}\, A)$ has $\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(G, Q) = 0$ for each G in $\mathcal{G}$. I must show $Q \cong 0$ in $\mathsf{K}(\mathsf{Pro}\, A)$.

First, I can consider construction 1.2 with M equal to A, viewed as an A-left-module. The corresponding complex P^* has suspensions $\Sigma^i P^*$, and by the construction of $\mathcal{G}$ each $\Sigma^i P^*$ is isomorphic to a complex in $\mathcal{G}$, so $\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(\Sigma^i P^*, Q)$ is zero. Hence

$$\begin{aligned} 0 &= \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(\Sigma^i P^*, Q) \\ &\cong \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(P^*, \Sigma^{-i} Q) \\ &\cong \mathrm{H}^0 \operatorname{Hom}_A(A, \Sigma^{-i} Q) \\ &\cong \mathrm{H}^{-i} Q, \end{aligned}$$

where the second $\cong$ is by lemma 1.5. So Q is exact.

Secondly, let me show that for each j, the j'th cycle module $\mathrm{Z}^j Q$ of Q is flat. It is clearly enough to do this for $\mathrm{Z}^0 Q$. I shall use the criterion of [2, chp. VI, exer. 6]. So suppose that $a_1, \ldots, a_m$ in A and $z_1, \ldots, z_m$ in $\mathrm{Z}^0 Q$ satisfy the relation

$$\sum_s a_s z_s = 0. \tag{1}$$

Consider the finitely generated submodule $M = Az_1 + \cdots + Az_m$ of $\mathrm{Z}^0 Q$. Since $\mathrm{Z}^0 Q$ is a submodule of Q^0, so is M, and as M is finitely generated while Q^0 is projective and A coherent, it follows that M is finitely presented. So M is among the modules considered in construction 1.2, and there is a corresponding complex P^*. As above, by the construction of $\mathcal{G}$ the complex P^* is isomorphic to a complex in $\mathcal{G}$, so $\mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(P^*, Q)$ is zero. Hence

$$0 = \mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(P^*, Q) \cong \mathrm{H}^0\mathrm{Hom}_A(M, Q)$$

by lemma 1.4.

So each homomorphism $M \longrightarrow Q^0$ for which the composition $M \longrightarrow Q^0 \longrightarrow Q^1$ is zero factors through $Q^{-1} \longrightarrow Q^0$. In other words, each homomorphism $M \longrightarrow \mathrm{Z}^0 Q$ factors through the canonical surjection $Q^{-1} \xrightarrow{\sigma} \mathrm{Z}^0 Q$. But M is a submodule of $\mathrm{Z}^0 Q$, so in particular the inclusion $M \hookrightarrow \mathrm{Z}^0 Q$ factors,

$$\begin{tikzcd} & M \arrow[dl, "f"'] \arrow[d, hook] \\ Q^{-1} \arrow[r, "\sigma"'] & \mathrm{Z}^0 Q. \end{tikzcd}$$

Applying f to $\sum_s a_s z_s = 0$ gives $\sum_s a_s f(z_s) = 0$ in Q^{-1}. But Q^{-1} is projective, hence flat, and so by [2, chp. VI, exer. 6] there exist $a_{11}, \ldots, a_{mn}$ in A and $q_1, \ldots, q_n$ in Q^{-1} so that

$$f(z_s) = \sum_t a_{st} q_t \tag{2}$$

and

$$\sum_s a_s a_{st} = 0. \tag{3}$$

Applying σ to equation (2) gives

$$z_s = \sum_t a_{st} \sigma(q_t). \tag{4}$$

However, when equation (1) implies the existence of $a_{11}, \ldots, a_{mn}$ in A and $\sigma(q_1), \ldots, \sigma(q_n)$ in $\mathrm{Z}^0 Q$ so that equations (3) and (4) are satisfied, then [2, chp. VI, exer. 6] says that $\mathrm{Z}^0 Q$ is flat as desired.

Finally, note that by remark 2.2 there is an integer N so that each flat A-left-module F has $\mathrm{pd}\, F \leq N$. Hence $\mathrm{pd}\, \mathrm{Z}^{j+N} Q \leq N$ for each j. But there is an exact sequence

$$0 \to \mathrm{Z}^j Q \to Q^j \to \cdots \to Q^{j+N-1} \to \mathrm{Z}^{j+N} Q \to 0,$$

and since $Q^j, \ldots, Q^{j+N-1}$ are projective there follows $\operatorname{pd} \mathrm{Z}^j Q \leq 0$, that is, $\mathrm{Z}^j Q$ is projective for each j.

So Q is an exact complex of projectives where each cycle module is also projective. Hence Q is split exact, and so in particular null homotopic, so $Q \cong 0$ in $\mathsf{K}(\mathsf{Pro}\, A)$ as desired. □

3. The subcategory of compact objects

Setup 3.1. In this section, A is again a coherent ring for which each flat A-left-module has finite projective dimension.

The compactly generated triangulated category $\mathsf{K}(\mathsf{Pro}\, A)$ has the full subcategory $\mathsf{K}(\mathsf{Pro}\, A)^{\mathrm{c}}$ of compact objects. And the derived category $\mathsf{D}(A^{\mathrm{op}})$ of A-right-modules has the full subcategory $\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$ of complexes with bounded cohomology consisting of finitely presented modules.

Theorem 3.2. *There is an equivalence of triangulated categories*

$$\mathsf{K}(\mathsf{Pro}\, A)^{\mathrm{c}} \xrightarrow{\simeq} \mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})^{\mathrm{op}}.$$

Proof. Consider again the set $\mathcal{G}$ from construction 2.3. Theorem 2.4 says that $\mathcal{G}$ is a set of compact generators for $\mathsf{K}(\mathsf{Pro}\, A)$.

Let C be the full subcategory of $\mathsf{K}(\mathsf{Pro}\, A)$ consisting of objects which are finitely built from objects G in $\mathcal{G}$. Let D be the full subcategory of $\mathsf{K}(\mathsf{Pro}\, A^{\mathrm{op}})$ consisting of objects which are finitely built from objects of the form G^* with G in $\mathcal{G}$.

Each object G in $\mathcal{G}$ is a complex of finitely generated projective modules, so the canonical chain maps $G \to G^{**}$ and $G^* \to G^{***}$ are isomorphisms. Hence

$$\mathsf{C} \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathsf{D}^{\mathrm{op}} \tag{5}$$

are quasi-inverse equivalences of triangulated categories. Indeed, let me show that this gives the equivalence stated in the theorem: First, the category C consists of the objects finitely built from a set of compact generators of the compactly generated triangulated category $\mathsf{K}(\mathsf{Pro}\, A)$, so C is equal to $\mathsf{K}(\mathsf{Pro}\, A)^{\mathrm{c}}$ by the Thomason Localization Theorem, [4, thm. 2.1].

Secondly, let me consider the category D. It consists of the objects finitely built from objects of the form G^* with G in $\mathcal{G}$. By the definition of $\mathcal{G}$, there is one object G in each isomorphism class of objects of the form $\Sigma^i P^*$ with P^* coming from construction 1.2. So up to isomorphism, there is one object G^* in each isomorphism class of objects of the form $\Sigma^j P$ with P coming from construction 1.2. Recall

from construction 1.2 that P is a projective resolution of the A-right-module M^* which comes from the finitely presented A-left-module M. It follows that D consists of the objects finitely built from projective resolutions of the form P.

Now, if D had consisted of the objects finitely built from projective resolutions of *all* finitely presented A-right-modules, then D would have been the subcategory of $\mathsf{K}(\mathsf{Pro}\, A^{\mathrm{op}})$ consisting of projective resolutions of all complexes with bounded finitely presented cohomology, and it is classical that this subcategory is equivalent to $\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$. So I would have been done: Equation (5) would have given the equivalence stated in the theorem.

As it is, D only consists of objects finitely built from projective resolutions P of A-right-modules of the form M^* with M a finitely presented A-left-module. However, this makes no difference because it turns out that I can finitely build the projective resolution of any finitely presented A-right-module from projective resolutions of the form P.

To see this, suppose that N is a finitely presented A-right-module, and let

$$Q = \cdots \to Q^{-2} \to Q^{-1} \to Q^0 \to 0 \to \cdots$$

be a projective resolution of N. Since all projective resolutions of N are isomorphic in $\mathsf{K}(\mathsf{Pro}\, A^{\mathrm{op}})$, I can suppose that Q consists of finitely generated projective A-right-modules.

Now

$$\widetilde{Q} = \cdots \to Q^{-4} \to Q^{-3} \to Q^{-2} \to 0 \to \cdots$$

is the double suspension of a projective resolution of $\mathrm{Z}^{-1}Q$, the (-1)'st cycle module of Q, and the complex Q is finitely built from Q^0 and Q^{-1} (viewed as complexes concentrated in degree zero) along with $\widetilde{Q}$.

Both Q^0 and Q^{-1} are projective resolutions of the form P, since they are both projective resolutions of modules of the form M^*, namely, they are resolutions of $(Q^{0*})^* \cong Q^0$ and $(Q^{-1*})^* \cong Q^{-1}$.

And $\widetilde{Q}$ is the double suspension of a projective resolution of the form P because $\mathrm{Z}^{-1}Q$ has the form M^* for a finitely presented A-left-module M. To see this, complete $Q^{0*} \to Q^{-1*}$ with its cokernel,

$$Q^{0*} \to Q^{-1*} \to M \to 0.$$

Here M is finitely presented and M^* sits in the exact sequence

$$0 \to M^* \to Q^{-1**} \to Q^{0**}.$$

But Q^0 and Q^{-1} are finitely generated, so up to isomorphism the last map here is just $Q^{-1} \to Q^0$, so up to isomorphism, the kernel M^* is just the kernel of $Q^{-1} \to Q^0$, that is, it is $\mathrm{Z}^{-1}Q$. So $\mathrm{Z}^{-1}Q$ has the form M^*. $\square$

4. The dualizing complex case

Setup 4.1. In this section, k is a field, A is a k-algebra which is left-coherent and right-noetherian, B is a left-noetherian k-algebra, and ${}_BD_A$ is a dualizing complex over B and A.

See [8, def. 1.1] for the definition of dualizing complexes.

Theorem 4.2. *There is an equivalence of triangulated categories*

$$\mathsf{K}(\mathsf{Pro}\,A)^{\mathrm{c}} \xrightarrow{\simeq} \mathsf{D}^{\mathrm{f}}(B).$$

Proof. Since there is a dualizing complex ${}_BD_A$ between B and A, each flat A-left-module has finite projective dimension by [3]. Moreover, A is clearly coherent. So section 3 applies to A, and theorem 3.2 gives an equivalence

$$\mathsf{K}(\mathsf{Pro}\,A)^{\mathrm{c}} \xrightarrow{\simeq} \mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})^{\mathrm{op}}.$$

But existence of ${}_BD_A$ gives an equivalence

$$\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\simeq} \mathsf{D}^{\mathrm{f}}(B)$$

by [8, prop. 1.3(2)], and composing the two equivalences proves the theorem. ∎

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